JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 15, No. 4, July–August 1992

Alternative Approach to the Solution of Lambert's Problem

Steven L. Nelson* and Paul Zarchan†
Charles Stark Draper Laboratory, Inc., Cambridge, Massachusetts 02139

A method for solving the two-point, two-body orbital transfer boundary-value problem, commonly referred to as Lambert's problem, is presented. Previous algorithms have depended heavily on the geometric properties of conic sections to obtain an iterative solution. An alternative approach is offered, making use of velocity and time functions of the flyout angle that have been derived directly from the equations of motion. A procedure is presented that rapidly iterates directly on the flyout angle until the desired initial velocity vector can be obtained.

Statement of Lambert's Problem

THE boundary-value problem for a two-point, two-body transfer in a Newtonian central gravity field (often referred to as Lambert's problem) can be stated as follows: given an initial position vector r_1 and a final position vector r_2, find the initial velocity vector V, which will allow a transfer from r_1 to r_2 in time t_f. Originally, this problem concerned tracking bodies that were orbiting the sun, but today it has application in problems such as spacecraft navigation and strategic missile guidance.

Ideally, given the parameters r_1, r_2, and μ (the gravitational parameter), the solution to Lambert's problem would be achieved with an equation that gives the velocity vector V as a function of the transfer time:

$$V = f(t_f) \tag{1}$$

For several centuries, the derivation of this function has proved to be intractable.

Background

Several iterative schemes have been developed that make clever use of geometric properties of conic sections, which are known to be the shape of the possible trajectories for the two-point, two-body transfer.[1] The classical iterative approach to Lambert's problem was first devised by Gauss.[2] Gauss used a successive substitution algorithm to find the sector-triangle ratio of the elliptical sector bounded by r_1 and r_2 for the desired orbit. From that result, the initial velocity vector could be computed using other known properties of an ellipse and the equations of motion.

Many other methods have been developed that iterate on various parameters of the conics that describe the desired trajectories.[3] This type of approach has been successfully extended to include all types of trajectories (elliptical, parabolic, and hyperbolic). In particular, Battin and Vaughan[4] were able to extend Gauss' method to include all types of trajectories and to remove the singularity in Gauss' method that existed when the range angle formed by r_1 and r_2 (the difference in the true anomalies, also called the central angle) was 180 deg.

The difficulty with all of these approaches is that none of the iterated parameters, or even the results of the iteration, are easy to intuitively relate to the obvious physical parameters of the trajectory. For example, Gauss' method iterates until the value of the sector-triangle ratio stops changing by a specified tolerance. The problem is that it is intuitively difficult (although, of course, computationally possible) to relate the tolerance in the sector-triangle ratio to a more useful value such as a tolerance in the time of flight.

New Approach

The new approach builds on closed-form solutions to the free-flight ballistics problem, which have been known for some time. More than three decades ago, Wheelon[5] was able to derive velocity and time equations as functions of the complement of the flyout angle (the flyout angle is the angle between the local horizontal at the initial position vector and the initial velocity vector, and it is also known as the flight-path or heading angle). Wheelon's velocity and time equations are not sufficient to solve Lambert's problem since both are functions of the flyout angle rather than time. Another function must therefore be derived that iterates the flyout angle until the transfer time is achieved. Also, since Wheelon was only concerned with elliptical (ballistic) trajectories, his time equation must be extended to include the parabolic and hyperbolic cases.

With the velocity, time, and iterator functions known, Lambert's problem can be solved quite simply. After an initial guess is made for the flyout angle, a corresponding initial value for the transfer time is found exactly using the time equation. If the transfer time thus found is not within a specified tolerance to the desired transfer time, then the iterator function is used to generate a new value for the flyout angle based on the results of the previous estimates. This new flyout angle produces a new corresponding flyout time, and the cycle repeats until the specified tolerance to the desired transfer time is met. The velocity function can then be used to immediately determine the required initial velocity vector using the final resulting flyout angle.

To be specific, if γ is the flyout angle, V the initial velocity vector, t the transfer time, t_f the desired (final) transfer time, $f_V(\gamma)$ the velocity function, $f_t(\gamma)$ the time function, and $h(\gamma_n, \gamma_{n-1}, \ldots, \gamma_{n-k+1}, t_n, t_{n-1}, \ldots, t_{n-k+1})$ the iterator function, then

$$V = f_V(\gamma) \tag{2}$$

$$t = f_t(\gamma) \tag{3}$$

$$\gamma_{n+1} = h(\gamma_n, \gamma_{n-1}, \ldots, \gamma_{n-k+1}, t_n, t_{n-1}, \ldots, t_{n-k+1}) \tag{4}$$

where k is the number of previous estimates required by h.

After an initial guess is made for γ, Eqs. (3) and (4) are iterated until t_n is within a specified tolerance of t_f or, alternately, the change in t_n (or γ_n) is within a specified tolerance. Equation (2) can then be used to immediately get the desired initial value for the velocity vector.

It should be noted that in the new approach the variable that is iterated (the flyout angle) and the variable used to terminate

Received April 20, 1990; revision received July 30, 1991; accepted for publication July 31, 1991.

*Member of the Technical Staff.

†Member of the Technical Staff. Associate Fellow AIAA.

the iteration (the time of flight) are both obvious physical parameters of the desired trajectory and are, therefore, intuitively related to the problem.

This paper will show that the time function is continuous and well behaved, so that only a relatively simple iterator function is required to provide rapid convergence to the desired transfer time, and thereby to the solution to Lambert's problem.

Review of the Velocity Equation

Dividing the velocity vector $\boldsymbol{V}$ into two components, we get

$$\boldsymbol{V} = (V, 1_V) \tag{5}$$

where V and 1_V are the magnitude and unit direction vector corresponding to $\boldsymbol{V}$. Since a Newtonian central-force gravity field is assumed, the trajectory under consideration must lie entirely in a plane determined by the initial and final position vectors,[6] $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$. If the chosen reference is the local horizontal at $\boldsymbol{r}_1$, which is on the same side of $\boldsymbol{r}_1$ as the intended trajectory (Fig. 1), then the unit direction vector 1_V is a function of the flyout angle γ.

If

$$\boldsymbol{i} = \frac{\boldsymbol{r}_1}{|\boldsymbol{r}_1|} \tag{6}$$

$$\boldsymbol{j} = \frac{(\boldsymbol{r}_1 \times \boldsymbol{r}_2) \times \boldsymbol{r}_1}{|(\boldsymbol{r}_1 \times \boldsymbol{r}_2) \times \boldsymbol{r}_1|} \tag{7}$$

then

$$1_V = \boldsymbol{i} \sin\gamma + \boldsymbol{j} \cos\gamma \tag{8}$$

Note that Eq. (7) is simply the equation for the horizontal reference for γ.

The magnitude V can also be shown to be a function of γ as follows. Consider the radial and transverse components of a vector $\boldsymbol{r}$ with polar coordinates (r, θ) that represent the position vector of a point on the desired trajectory (Fig. 1). The position is given by

$$\boldsymbol{r} = r\,1_r \tag{9}$$

the velocity is

$$\dot{\boldsymbol{r}} = \dot{r} 1_r + r\dot{\theta} 1_\theta \tag{10}$$

and the acceleration is

$$\ddot{\boldsymbol{r}} = \left(\ddot{r} - r(\dot{\theta})^2\right) 1_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta}) 1_\theta \tag{11}$$

Given that the only force acting is gravity, then

$$\ddot{\boldsymbol{r}} = -\frac{\mu}{r^2} 1_r \tag{12}$$

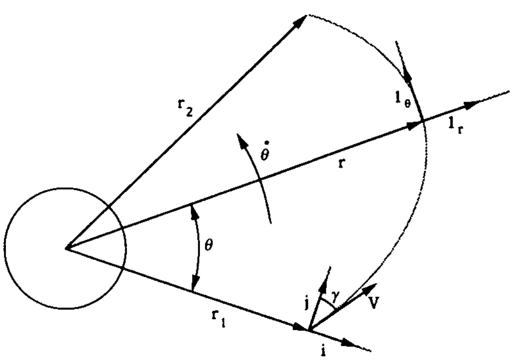

Fig. 1 Geometry of Lambert's problem.

Substituting into Eq. (11) and separating radial and transverse components

$$\ddot{r} - r(\dot{\theta})^2 + \frac{\mu}{r^2} = 0 \tag{13}$$

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\theta}) = 0 \tag{14}$$

Equation (14) allows us to conclude that angular momentum is conserved and allows us to substitute the initial moment of momentum per unit mass P at $\boldsymbol{r}_1$:

$$r^2\dot{\theta} = P = |\boldsymbol{r}_1| V \cos\gamma \tag{15}$$

By using Eq. (15) to change the independent variable from time t to angle θ and making the variable substitution of $u = 1/r$, Eq. (13) may be rewritten as[5]

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} + u = \frac{1}{\lambda |\boldsymbol{r}_1| \cos^2\gamma} \tag{16}$$

where

$$\lambda = \frac{|\boldsymbol{r}_1| V^2}{\mu} \tag{17}$$

Solving, we obtain

$$u(\theta) = \frac{1}{r(\theta)} = \frac{1}{\lambda|\boldsymbol{r}_1| \cos^2\gamma} - \frac{\tan\gamma}{|\boldsymbol{r}_1|}\sin\theta + \frac{1}{|\boldsymbol{r}_1|}\left[1 - \frac{1}{\lambda\cos^2\theta}\right]\cos\theta \tag{18}$$

Simplifying,

$$\frac{|\boldsymbol{r}_1|}{r(\theta)} = \frac{1-\cos\theta}{\lambda\cos^2\gamma} + \frac{\cos(\theta+\gamma)}{\cos\gamma} \tag{19}$$

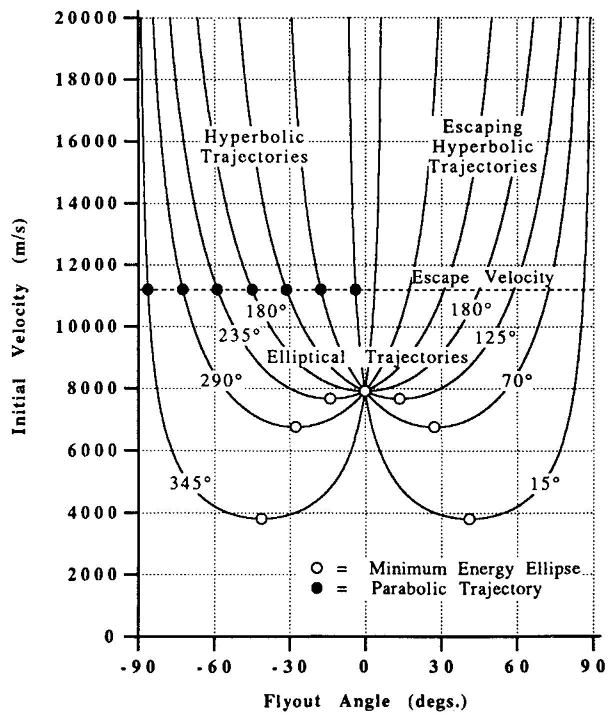

Fig. 2 Velocity vs flyout angle.

To determine an expression for the velocity V, recall the definition of λ and substitute

$$\frac{|r_1|}{r(\theta)} = \frac{1-\cos\theta}{[(|r_1|V^2)/\mu]\cos^2\gamma} + \frac{\cos(\theta+\gamma)}{\cos\gamma}$$

$$V = \sqrt{\frac{\mu}{|r_1|}\frac{r(\theta)(1-\cos\theta)}{|r_1|\cos^2\gamma - r(\theta)\cos(\theta+\gamma)\cos\gamma}} \tag{20}$$

When $\theta=\theta_f$ (the range angle between r_1 and r_2), this becomes

$$V = \sqrt{\frac{\mu}{|r_1|}\frac{|r_2|(1-\cos\theta_f)}{|r_1|\cos^2\gamma - |r_2|\cos(\theta_f+\gamma)\cos\gamma}} \tag{21}$$

θ_f is related to r_1 and r_2 as follows:

$$\theta_f = \cos^{-1}\left(\frac{r_1\cdot r_2}{|r_1|\,|r_2|}\right) \tag{22}$$

The value for the arccosine function in Eq. (22) depends on whether the range angle is greater or less than 180 deg.

Since θ_f is a function of r_1 and r_2, Eq. (21) along with Eq. (8) completes the function of velocity with respect to the flyout angle $[f_V(\gamma)]$. Figure 2 is a plot of the magnitude of velocity vs flyout angle for several possible range angles. In this example, the magnitudes of r_1 and r_2 are both equal to one Earth radius, and the gravity field is that of Earth, with a point mass assumed.

Note that there are generally two values for the flyout angle γ that correspond to a particular initial velocity magnitude V. For values of V greater than the escape velocity, only the smaller of the two values of γ will actually result in the desired orbital transfer. This fact will be useful in bounding the possible values of the flyout angle, which in turn will assist in the development of the iterator function. For values of V equal to the escape velocity, the lower value of γ corresponds to the parabolic trajectory from r_1 to r_2. The minimum value of V corresponds to the minimum energy trajectory.

Extension of the Time of Flight Equation

An expression for the time of flight as a function of the range angle θ_f can be found by referring to Eqs. (15) and (19):

$$r^2\dot\theta = |r_1|V\cos\gamma$$

$$\left\{\frac{|r_1|}{[(1-\cos\theta)/\lambda\cos^2\gamma)] + [\cos(\theta+\gamma)/\cos\gamma]}\right\}^2\frac{d\theta}{dt}$$

$$= |r_1|V\cos\gamma$$

$$\frac{|r_1|}{V\cos\gamma}\left(\frac{1-\cos\theta}{\lambda\cos^2\gamma} + \frac{\cos(\theta+\gamma)}{\cos\gamma}\right)^{-2}d\theta = dt$$

$$\frac{|r_1|}{V\cos\gamma}\int_0^{\theta(t)}\left(\frac{1-\cos\theta}{\lambda\cos^2\gamma} + \frac{\cos(\theta+\gamma)}{\cos\gamma}\right)^{-2}d\theta = t \tag{23}$$

Simplifying the integrand produces

$$t = \frac{|r_1|}{V\cos\gamma}\int_0^{\theta(t)} \times \frac{\lambda^2\cos^4\gamma}{[1+(\lambda\cos^2\gamma-1)\cos\theta - \lambda\cos\gamma\sin\gamma\sin\theta]^2}d\theta \tag{24}$$

The solution to this type of integral is dependent on the values of the coefficients of the denominator of the integrand.[7] If

$$a = 1$$

$$b = \lambda\cos^2\gamma - 1$$

$$c = -\lambda\cos\gamma\sin\gamma$$

then there are three domains for the solution of the integral corresponding to

$$a^2 > b^2 + c^2$$

$$a^2 < b^2 + c^2$$

$$a^2 = b^2 + c^2$$

or, alternatively,

$$0 > \lambda\cos^2\gamma(\lambda-2) \rightarrow \lambda < 2$$

$$0 < \lambda\cos^2\gamma(\lambda-2) \rightarrow \lambda > 2$$

$$0 = \lambda\cos^2\gamma(\lambda-2) \rightarrow \lambda = 2$$

When the integral is solved for each of these domains, the following solution for the time of flight is obtained:

$0<\lambda<2$:

$$t = \frac{|r_1|}{V\cos\gamma}\left\{\frac{\tan\gamma(1-\cos\theta)+(1-\lambda)\sin\theta}{(2-\lambda)\{[(1-\cos\theta/\lambda\cos^2\gamma] + [\cos(\theta+\gamma)/\cos\gamma]\}} + \frac{2\cos\gamma}{\lambda[(2/\lambda)-1]^{3/2}}\tan^{-1}\frac{[(2/\lambda)-1]^{1/2}}{\cos\gamma\cot(\theta/2)-\sin\gamma}\right\} \tag{25a}$$

$\lambda>2$:

$$t = \frac{|r_1|}{V\cos\gamma}\left\{\frac{\tan\gamma(1-\cos\theta)+(1-\lambda)\sin\theta}{(2-\lambda)\{[(1-\cos\theta)/\lambda\cos^2\gamma] + [\cos(\theta+\gamma)/\cos\gamma]\}} - \frac{\cos\gamma}{\lambda[1-2/\lambda]^{3/2}}\ln\frac{\sin\gamma-\cos\gamma\cot(\theta/2)-[1-2/\lambda]^{1/2}}{\sin\gamma-\cos\gamma\cot(\theta/2)+[1-2/\lambda]^{1/2}}\right\} \tag{25b}$$

$\lambda = 2$:

$$t = \frac{2|r_1|}{3V}\left[\frac{3\cos\gamma\cot(\theta/2)}{[\cos\gamma\cot(\theta/2)-\sin\gamma]^2} + \frac{1}{[\cos\gamma\cot(\theta/2)-\sin\gamma]^3}\right] \tag{25c}$$

For the total time of flight t_f, Eqs. (25a–c) can be further simplified using Eq. (19):

$0<\lambda<2$:

$$t_f = \frac{|r_1|}{V\cos\gamma}\left\{\frac{\tan\gamma(1-\cos\theta_f)+(1-\lambda)\sin\theta_f}{(2-\lambda)|r_1|/|r_2|} + \frac{2\cos\gamma}{\lambda[(2/\lambda)-1]^{3/2}}\tan^{-1}\frac{[(2/\lambda)-1]^{1/2}}{\cos\gamma\cot(\theta_f/2)-\sin\gamma}\right\} \tag{26a}$$

$\lambda>2$:

$$t_f = \frac{|r_1|}{V\cos\gamma}\left\{\frac{\tan\gamma(1-\cos\theta_f)+(1-\lambda)\sin\theta_f}{(2-\lambda)|r_1|/|r_2|} - \frac{\cos\gamma}{\lambda[1-2/\lambda]^{3/2}}\ln\frac{\sin\gamma-\cos\gamma\cot(\theta_f/2)-[1-2/\lambda]^{1/2}}{\sin\gamma-\cos\gamma\cot(\theta_f/2)+[1-2/\lambda]^{1/2}}\right\} \tag{26b}$$

$\lambda = 2$:

$$t_f = \frac{2|r_1|}{3V}\left[\frac{3\cos\gamma\cot(\theta_f/2)}{[\cos\gamma\cot(\theta_f/2)-\sin\gamma]^2} + \frac{1}{[\cos\gamma\cot(\theta_f/2)-\sin\gamma]^3}\right] \tag{26c}$$

Although no knowledge of the elliptic, hyperbolic, or parabolic nature of the trajectories was used to solve for the equation for the time of flight, it is of interest to note that the separate domains of convergence for Eq. (24) correspond exactly to each of those different trajectories.[1] Equations (26b) and (26c) are the extensions required to complete the time function [$f_t(\gamma)$] since they correspond to the hyperbolic and parabolic transfer orbits, respectively.

Much of the past work on Lambert's problem has involved finding a single equation (usually in the form of a power series or continued fraction expansion) that combines the elliptic, parabolic, and hyperbolic cases. This was generally done to avoid numerical difficulties near the parabolic case. In particular, the method used in the Apollo guidance computer[8] used just such a universal solution. Given modern computer technology, numerical difficulties are much less of a problem today since higher precision used in modern calculations considerably narrows the range of values for λ near 2 that causes problems. For all values of λ not within that range, Eqs. (26a) and (26b) can be used. For values of λ near 2, Eq. (26c) will provide a very adequate approximation for the time of flight. Although outside the scope of this paper, if a universal solution is still desired, possible approaches to achieving it could involve solving the integral in Eq. (24) using complex variables, or expanding the integrand into a power series and then integrating it term by term. Of course, care must be exercised to insure that the resulting series converges for all range angles.

Figure 3 is a plot of flight times for various range angles vs the flyout angle γ for a particular pair of position vectors and a selected gravity field. As with Fig. 2, for this example the magnitudes of r_1 and r_2 are both equal to one Earth radius, and the gravity field is that of Earth, with a point mass assumed.

It should be noted that the time of flight is a continuous, monotonically increasing function of the flyout angle. This will be important in the development of the iterator function in the following.

Iterator Function

To complete the set of equations required to allow an iterative solution to the Lambert problem, an iterator function must be found. In this case, the task of the iteration is to numerically find a root of

$$t_f = f_t(\gamma) \tag{27}$$

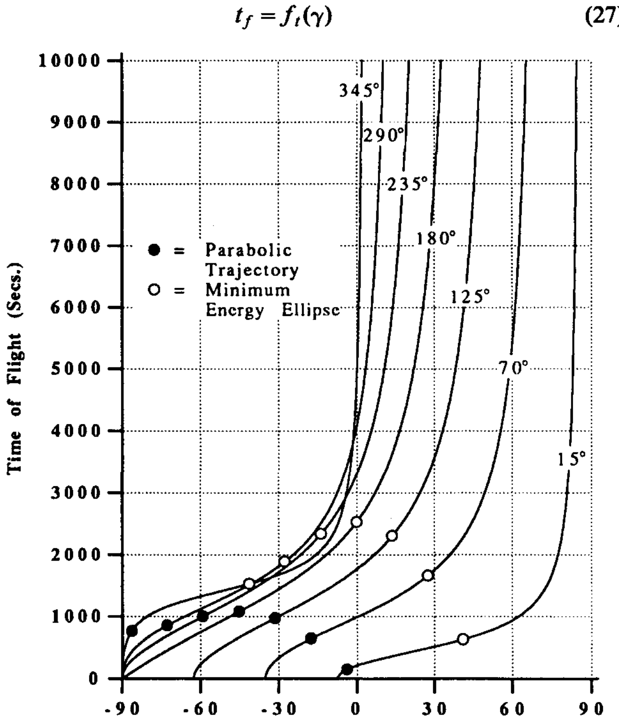

Fig. 3 Time of flight vs flyout angle.

Finding a root of an equation numerically is often a difficult task at best. Fortunately, the time function $f_t(\gamma)$ is a continuous, monotonically increasing function of γ. This prevents the numerical difficulties associated with local maxima and minima. Also, the range of possible values for γ is bounded by $\gamma_{\min}$ and $\gamma_{\max}$.

Figure 3 helps to infer the reasons for the flyout angle being bounded. The lower bound $\gamma_{\min}$ for each range angle corresponds to the most direct (zero transfer time) trajectory. For range angles of greater than 180 deg, this is a degenerate hyperbola (two straight lines), which is the limiting case of the hyperbolic trajectories from r_1 to r_2 as the distance of the nearest approach to the gravitational point source goes to zero. This always corresponds to a flyout angle of -90 deg. For range angles of less than 180 deg, the zero transfer time trajectory is a straight line passing through the endpoints of r_1 and r_2. The corresponding flyout angle can be determined from Eq. (21) by noting that it occurs when the velocity V goes to infinity or, alternatively, when the denominator under the radical goes to zero. Thus,

$$|r_1| \cos^2 \gamma_{\min} - |r_2| \cos(\theta_f + \gamma_{\min}) \cos \gamma_{\min} = 0 \tag{28a}$$

$$\frac{|r_1|}{|r_2|} \cos \gamma_{\min} = \cos(\theta_f + \gamma_{\min}) \tag{28b}$$

$$\frac{|r_1|}{|r_2|} \cos \gamma_{\min} = \cos \theta_f \cos \gamma_{\min} - \sin \theta_f \sin \gamma_{\min} \tag{28c}$$

$$\frac{|r_1|}{|r_2|} = \cos \theta_f - \sin \theta_f \tan \gamma_{\min} \tag{28d}$$

$$\gamma_{\min} = \tan^{-1}\left[\frac{\cos \theta_f - (|r_1|/|r_2|)}{\sin \theta_f}\right] \tag{28e}$$

Equation (28e) is the equation for the minimum flyout angle $\gamma_{\min}$.

It is also true that for each pair of vectors r_1 and r_2 and their corresponding range angle there is a maximum value for the flyout angle $\gamma_{\max}$. This maximum bound arises from the fact that, as the flyout angle γ increases toward $\gamma_{\max}$, the initial velocity V also increases until the escape velocity is reached. Escape velocity V_e at the initial position vector is given by[1]

$$V_e = \sqrt{\frac{2\mu}{|r_1|}} \tag{29}$$

If Eq. (21) is examined with Eq. (29) in mind, it can be seen that

$$\frac{|r_2|(1-\cos \theta_f)}{|r_1| \cos^2 \gamma_{\max} - |r_2| \cos(\theta_f + \gamma_{\max}) \cos \gamma_{\max}} = 2 \tag{30}$$

After some manipulation, the preceding equation (30) becomes

$$\frac{1-\cos \theta_f}{(|r_1|/|r_2|) - \cos \theta_f + \sin \theta_f \tan \gamma_{\max}} = \frac{2}{1+\tan^2 \gamma_{\max}} \tag{31}$$

Equation (31) can be solved for $\tan \gamma_{\max}$ to get

$$\tan \gamma_{\max} = \frac{-\sin \theta_f \pm \sqrt{[2|r_1|/|r_2|](1-\cos \theta_f)}}{\cos \theta_f - 1} \tag{32}$$

The smaller root of this equation gives the value of the flyout angle, which corresponds to the parabolic trajectory. This smaller root can be substituted into Eq. (26c) to remove the flyout angle from the equation for time of flight in the parabolic case. Also, this smaller root bounds the values of the flyout angle that correspond to elliptical trajectories and can therefore be used for $\gamma_{\min}$ when only elliptical trajectories are

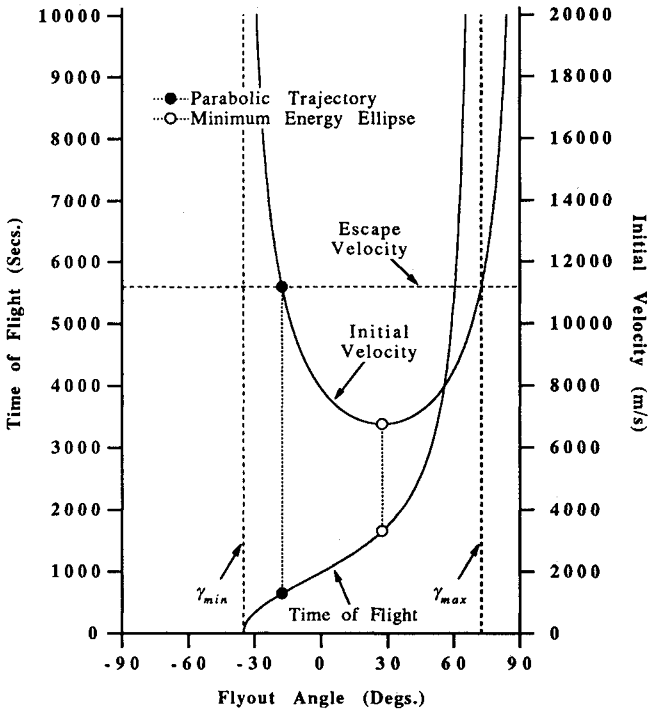

Fig. 4 Minimum and maximum flyout angles.

being considered. The larger root gives us the maximum flyout angle that is sought:

$$\gamma_{\max} = \tan^{-1}\left(\frac{\sin\theta_f + \sqrt{(2|r_1|/|r_2|)(1-\cos\theta_f)}}{1-\cos\theta_f}\right) \tag{33}$$

Figure 4 presents the velocity and time of flight equations on the same graph, indicating how $\gamma_{\min}$ and $\gamma_{\max}$ are related to the minimum flight time and escape velocity, respectively. As with previous examples, the magnitudes of r_1 and r_2 are both equal to one Earth radius, and the gravity field is that of Earth, with a point mass assumed. The range angle used for this example is 70 deg.

The fact that the time function increases monotonically with the flyout angle suggests that even a brute-force iterator function is sufficient to find a root of Eq. (27). In particular, Zarchan[9] was able to get acceptable results using the iterator function

$$\gamma_0 = -90 \text{ deg} \tag{34a}$$

and for $n > 0$:

$$\gamma_{n+1} = \gamma_n + \Delta\gamma \tag{34b}$$

where $\Delta\gamma$ is a specified tolerance on the flyout angle (also implying a tolerance on the time of flight). The iteration terminates when $t_n[=f_t(\gamma_n)] > t_f$.

A more efficient method of obtaining the flyout angle would make use of a greater number of the previous estimates (γ_n, t_n). Care must be exercised to avoid using too many of these previous estimates since the improvement in the number of iterations required to converge on a solution can become offset by the additional time required in each iteration to perform a more complicated iterator function. For this reason, and for the sake of simplicity, a two-point (two-estimate) iterator function will be used to investigate how the new approach can be applied to Lambert's problem. This is not intended to imply that the iterator function presented is the best that is possible. It does indicate, however, that even a simple iterator function will give good performance, and it also illustrates in a generic manner how an alternative iterator might be used.

Using the form for the iterator function of Eq. (4) for two points ($k = 2$):

$$\gamma_{n+1} = h(\gamma_n, \gamma_{n-1}, t_n, t_{n-1}) \tag{35}$$

A modification of a numerical algorithm known as the secant method[10] along with the computed values for $\gamma_{\min}$ and $\gamma_{\max}$ allow a specific formulation of the iterator to be developed:

$$\gamma_0 = \frac{\gamma_{\min} + \gamma_{\max}}{2}, \qquad t_0 = f_t(\gamma_0) \tag{36a}$$

if $t_0 > t_f$ then

$$\gamma_1 = \frac{\gamma_{\min} + \gamma_0}{2}, \qquad t_1 = f_t(\gamma_1) \tag{36b}$$

otherwise

$$\gamma_1 = \frac{\gamma_{\max} + \gamma_0}{2}, \qquad t_1 = f_t(\gamma_1) \tag{36c}$$

and for $n > 1$:

$$\gamma_{n+1} = \gamma_n + \frac{(\gamma_n - \gamma_{n-1})(t_f - t_n)}{(t_n - t_{n-1})}, \qquad t_n = f_t(\gamma_n) \tag{36d}$$

At each iteration, γ_{n+1} is checked to insure that $\gamma_{\min} < \gamma_{n+1} < \gamma_{\max}$. If not, then an alternate function is used:

$$\gamma_{n+1} = \frac{\mathrm{MAX}(\gamma_n, \gamma_{n-1}) + \gamma_{\max}}{2} \tag{36e}$$

or

$$\gamma_{n+1} = \frac{\mathrm{MIN}(\gamma_n, \gamma_{n-1}) + \gamma_{\min}}{2} \tag{36f}$$

where MAX and MIN are functions that select the greatest and least of their arguments, respectfully. The alternate iterator function insures that each new estimate of the flyout angle remains within the range from $\gamma_{\min}$ to $\gamma_{\max}$. The iteration terminates when t_n is within a specified tolerance of t_f. Note that from Eqs. (26a–c) V_n must be computed at each iteration to obtain t_n.

Using the New Approach to Solve Lambert's Problem

Having established the equations required to solve Lambert's problem $[f_V(\gamma), f_t(\gamma)$, and $h(\gamma_n, \gamma_{n-1}, t_n, t_{n-1})]$ as equations (21), (26a–c), and (36a–f), an example of their use to solve a particular problem will now be explored.

The parameters for the example are an Earth gravity field (again with a point source assumed), initial and final position vectors each of length equal to one Earth radius, a range angle of 45 deg, and a desired time of flight equal to 1600 s. The iteration will terminate when $t_n = t_f$ to eight significant digits. $\gamma_{\min}$ and $\gamma_{\max}$ are computed to be

$$\gamma_{\min} = -22.50000 \tag{37}$$

$$\gamma_{\max} = 78.75000 \tag{38}$$

The iteration is performed to produce

$\gamma_0 = 28.12500$ deg	$V_0 = 5.92414$ km/s	$t_0 = 1114.04545$ s
$\gamma_1 = 53.43750$ deg	$V_1 = 6.43467$ km/s	$t_1 = 1988.07749$ s
$\gamma_2 = 42.19854$ deg	$V_2 = 5.97688$ km/s	$t_2 = 1476.48157$ s
$\gamma_3 = 44.91205$ deg	$V_3 = 6.04906$ km/s	$t_3 = 1572.32493$ s
$\gamma_4 = 45.69558$ deg	$V_4 = 6.07404$ km/s	$t_4 = 1602.46686$ s
$\gamma_5 = 45.63145$ deg	$V_5 = 6.07193$ km/s	$t_5 = 1599.95389$ s
$\gamma_6 = 45.63263$ deg	$V_6 = 6.07197$ km/s	$t_6 = 1599.99992$ s
$\gamma_7 = 45.63263$ deg	$V_7 = 6.07197$ km/s	$t_7 = 1600.00000$ s

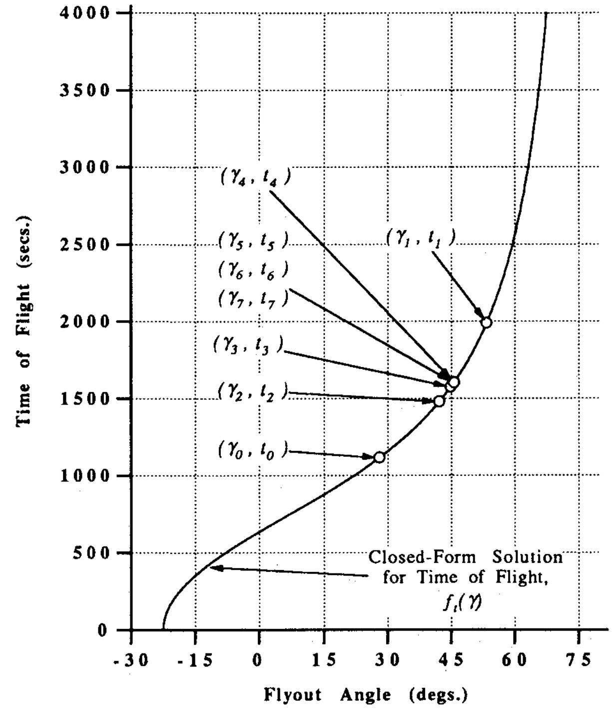

Fig. 5 Convergence to flyout angle.

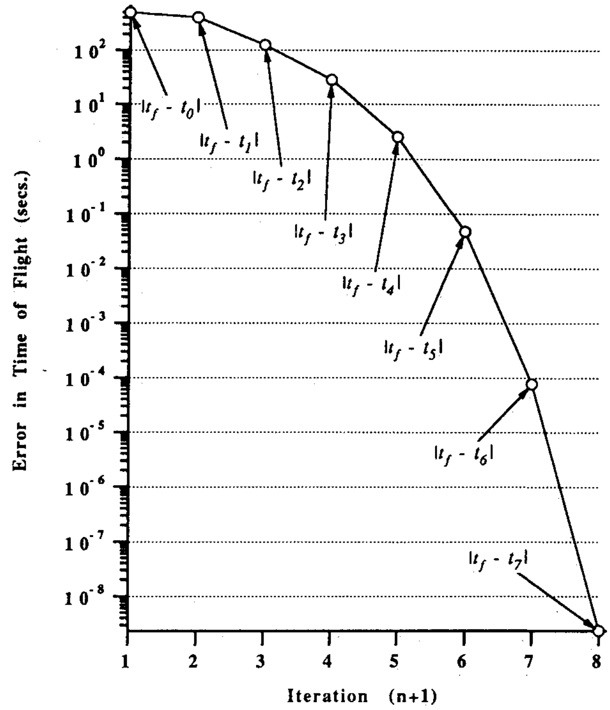

Fig. 6 Speed of convergence.

Table 1 Number of iterations required with $r_1 = r_2$[a]

	t_f, s										
θ_f, deg	50	100	200	400	800	1600	3200	6400	12,800	25,600	51,200
5	11	9	5	7	7	12	12	13	14	15	16
30	14	12	9	6	6	7	10	12	12	11	13
55	14	12	11	7	5	8	12	10	14	12	11
80	15	13	11	7	6	7	8	9	10	10	12
105	15	13	11	6	6	6	6	7	9	11	14
130	14	12	9	7	6	7	7	8	10	11	13
155	14	12	7	7	6	7	7	8	10	11	12
180	6	6	6	6	5	6	7	8	10	11	13
205	14	12	7	7	6	7	7	8	10	11	12
230	15	13	11	7	6	7	7	8	10	11	13
255	15	13	11	8	6	7	6	7	9	11	14
280	16	14	12	10	6	6	7	9	9	10	12
305	17	15	13	10	7	7	10	13	13	12	11
330	17	15	13	11	10	7	7	11	11	10	13
355	20	18	16	14	13	7	11	13	14	15	16

[a]Dashed lines indicate the separation between hyperbolic and elliptical trajectories, with hyperbolic trajectories on the left.

Table 2 Number of iterations required with $r_1 = 2r_2$[a]

	t_f, s										
θ_f, deg	50	100	200	400	800	1600	3200	6400	12,800	25,600	51,200
5	17	15	14	12	8	12	14	15	15	15	17
30	15	13	11	9	5	8	13	15	14	13	13
55	14	12	10	10	5	8	13	11	14	13	11
80	14	12	10	8	5	8	7	9	10	12	16
105	15	13	11	8	5	8	9	10	11	14	12
130	14	12	10	7	5	8	12	10	13	12	10
155	13	11	9	7	5	8	11	13	12	11	11
180	9	9	8	7	5	8	11	13	13	11	11
205	13	11	9	7	5	8	11	13	12	11	11
230	15	12	10	7	5	8	12	10	13	12	10
255	15	13	11	8	4	8	9	10	11	14	12
280	15	13	11	10	5	8	7	9	10	12	16
305	17	15	13	10	6	7	11	11	14	13	11
330	18	17	14	11	8	10	11	14	14	13	13
355	21	19	17	14	10	11	12	15	15	15	17

[a]Dashed lines indicate the separation between hyperbolic and elliptical trajectories, with hyperbolic trajectories on the left.

Table 3 Number of iterations required with $r_1 = r_2/2$[a]

θ_f, deg	t_f, s 50	100	200	400	800	1600	3200	6400	12,800	25,600	51,200
5	15	13	9	8	6	7	7	9	10	11	13
30	15	13	7	8	6	7	8	9	10	12	11
55	15	13	10	8	7	7	8	10	11	10	10
80	16	14	12	8	7	6	8	9	9	8	11
105	16	14	9	8	7	6	8	8	8	9	11
130	15	13	10	8	7	6	8	8	8	10	11
155	12	10	9	8	6	6	8	8	8	10	11
180	7	7	7	6	6	7	8	9	9	9	11
205	11	10	9	8	6	6	8	8	8	10	11
230	16	14	9	8	7	6	8	8	8	10	11
255	16	14	12	8	7	6	8	8	8	9	11
280	17	15	13	11	7	5	8	8	9	8	11
305	18	16	14	12	7	5	8	9	10	10	10
330	17	15	13	11	9	6	8	11	13	12	11
355	18	16	14	12	9	6	8	9	10	11	12

[a]Dashed lines indicate the separation between hyperbolic and elliptical trajectories, with hyperbolic trajectories on the left.

Figure 5 shows graphically how the iteration proceeds. Each estimate (γ_n, t_n) lies on the curve representing the time of flight as a function of the flyout angle. Convergence is rapid, especially after the first few estimates (Fig. 6).

With γ and V known, $\boldsymbol{V}$ can be found using Eqs. (6–8):

$$\boldsymbol{V} = V \mathbf{1}_V$$

$$\boldsymbol{V} = V(\boldsymbol{i} \sin \gamma + \boldsymbol{j} \cos \gamma)$$

$$\boldsymbol{V} = 6.07197\left[\boldsymbol{i} \sin (45.63263 \text{ deg}) + \boldsymbol{j} \cos (45.63263 \text{ deg})\right]$$

$$\boldsymbol{V} = (4.34068\boldsymbol{i} + 4.24586\boldsymbol{j}) \text{ km/s}$$

where $\boldsymbol{i}$ and $\boldsymbol{j}$ are defined as in Eqs. (6) and (7).

The two-point method offers an improvement over the brute-force method[9] by several orders of magnitude. The two-point method works well since the slope of the time function changes slowly with respect to the flyout angle, i.e., the second derivative of $f_t(\gamma)$ is generally small.

Performance of the Algorithm

To examine the performance of the algorithm, an Earth gravity field was once again chosen, with a point mass assumed. Three cases were examined: $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ both equal to one Earth radius; $\boldsymbol{r}_1$ equal to one Earth radius and $\boldsymbol{r}_2$ equal to one-half Earth radius: and $\boldsymbol{r}_1$ equal to one Earth radius and $\boldsymbol{r}_2$ equal to two Earth radii. The number of iterations of the algorithm required to obtain t_f to eight significant figures (using the two-point method outlined earlier) for various values of the range angle and t_f are recorded in Tables 1–3. All times of flight are in seconds.

Two important features of the new algorithm are evident from the tables. First, the new algorithm converges everywhere and is singular only at $\theta_f = 0$ or 360 deg. This is an improvement over the method derived by Gauss (which failed to converge for most hyperbolic orbits and was singular for $\theta_f = 180$ deg) and maintains the recent improvements in convergence and removal of the singularity made by Battin and Vaughan.[4]

The second point to be made concerns the efficiency of the algorithm. Implemented on an Apple Macintosh IIci personal computer, the algorithm as described in this paper typically required 435 μs per iteration. This corresponds to approximately 4.4 ms for the example presented. Thus, even though only a simple two-point iterator function has been used, the algorithm offers rapid performance for most applications.

Conclusions

The algorithm that has been presented offers a novel approach to Lambert's problem. It is derived directly from the equations of motion and requires no knowledge of the geometry of conic sections to derive either the iterated parameter (the flyout angle) or the variable used to terminate the iteration (typically the time of flight). In addition, the intuitive nature of the parameters involved with the new approach, the simple nature of the iteration, and the rapid convergence of the algorithm should make it attractive to a large community of users.

References

[1]Thomson, W. T., *Introduction to Space Dynamics,* Dover, New York, 1986.

[2]Gauss, K. F., *Theory of Motion of the Heavenly Bodies Moving about the Sun in Conic Sections,* translation of *Theoria Motus,* C. H. Davis, Dover, New York, 1963.

[3]Escobal, P. R., *Methods of Orbit Determination,* Wiley, New York, 1965.

[4]Battin, R. H., and Vaughan, R. M., "An Elegant Lambert Algorithm," *Journal of Guidance, Control, and Dynamics,* Vol. 7, No. 6, 1984, pp. 662–670.

[5]Wheelon, A. D., "Free Flight of a Ballistic Missile," *ARS Journal,* Vol. 29, No. 12, 1959, pp. 915–926.

[6]Thomas, G. B., Jr., *Calculus and Analytic Geometry,* 4th ed., Addison-Wesley, Reading, MA, 1972.

[7]Gradshteyn, I. S., and Ryshik, I. M., *Table of Integrals, Series, and Products,* translated form the Russian by Scripta Technical, Inc., Academic Press, New York, 1965, pp. 149–150.

[8]Marscher, W., "A Unified Method of Generating Conic Sections," Apollo Guidance, Navigation and Control, Massachusetts Inst. of Technology Instrumentation Laboratory, Cambridge, MA, Memorandum R-479, Feb. 1965.

[9]Zarchan, P. *Tactical and Strategic Missile Guidance,* Vol. 124, Progress in Astronautics and Aeronautics, AIAA, Washington, DC, 1990.

[10]Acton, F. S., *Numerical Methods that Work,* Harper & Row, New York, 1970.